

Ergodic Theory - Week 11

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1 Spectral theory of measure-preserving systems

- P1. (a)** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$ be an eigenfunction with eigenvalue $e(\alpha)$ for some $\alpha \in [0, 1)$. Calculate the spectral measure μ_f of f .

Solution: For any $n \in \mathbb{N}$, we use the spectral theorem to get

$$\begin{aligned} \int_0^1 e(nx) d\mu_f(x) &= \int_X T^n f \cdot \bar{f} d\mu = \int_X e(n\alpha) f \cdot \bar{f} d\mu = e(n\alpha) \|f\|_2^2 \\ &= \int_0^1 e(nx) \|f\|_2^2 d\delta_\alpha(x). \end{aligned}$$

Since linear combinations of trigonometric polynomials are dense in the space of continuous function, we deduce that

$$\int g(x) d\mu_f(x) = \int g(x) d(\|f\|_2^2 \delta_\alpha) \quad (1)$$

for all continuous functions g . Approximating the characteristic function of an open interval by a continuous function, we get that (1) holds when g is the characteristic function of an interval. We deduce that $\mu_f(A) = \|f\|_2^2 \delta_\alpha(A)$ for all open intervals, which implies that this also holds for all Borel sets (since their σ -algebra is generated by open intervals). We conclude that $\mu_f = \|f\|_2^2 \delta_\alpha(x)$.

- P2.** Find an eigenfunction and a weak-mixing function for the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T)$, where $T(x, y) = (x + \alpha, x + y)$, for some $\alpha \in [0, 1)$. Do the same for the system on the same space but with the map $S(x, y) = (x + \alpha, 2y)$.

Solution: Notice that for all $n \geq 0$,

$$T^n(x, y) = \left(x + n\alpha, y + nx + \frac{n(n-1)}{2}\alpha \right).$$

We consider the functions $f(x, y) = e(x)$ and $g(x, y) = e(y)$. It is easy to see that f is an eigenfunction:

$$Tf(x) = e(x + \alpha) = e(\alpha)e(x).$$

On the other hand, g is weak mixing since for any $n \neq 0$, we have

$$\left| \int_{\mathbb{T}^2} T^n g \cdot \bar{g} dm_{\mathbb{T}^2} \right| = \left| \int_{\mathbb{T}^2} e \left(nx + y + \frac{n(n-1)}{2}\alpha \right) e(-y) d(m_{\mathbb{T}} \times m_{\mathbb{T}})(x, y) \right| = \left| \int_0^1 e(nx) dx \right| = 0$$

and the claim follows by averaging over all $n \in \mathbb{N}$.

An entirely similar calculation shows that the same functions f, g are an eigenfunction and a weak-mixing function respectively for the second system.

P3. Given a measure-preserving system $(X, \mathcal{B}, \mu, T)$, show that for any $f, g \in L^2(X)$ with f weak-mixing, $f \otimes g \in L^2(X \times X)$ is weak-mixing.

Let $f, g \in L^2(X)$ and suppose that f is weak mixing. For any $N \in \mathbb{N}$, using Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_{X \times X} (T \times T)^n (f \otimes g) \cdot \overline{(f \otimes g)} \, d(\mu \times \mu) \right| &= \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_X T^n f \cdot \bar{f} \, d\mu \right| \left| \int_X T^n g \cdot \bar{g} \, d\mu \right| \\ &\leq \|g\|_2^2 \cdot \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_X T^n f \cdot \bar{f} \, d\mu \right|. \end{aligned}$$

By taking the limit as $N \rightarrow \infty$, since f is weak mixing, it follows that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_{X \times X} (T \times T)^n (f \otimes g) \cdot \overline{(f \otimes g)} \, d(\mu \times \mu) \right| = 0,$$

showing that $f \otimes g$ is weak mixing with respect to $T \times T$.

P4. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and consider the space

$$\mathcal{H}_c = \overline{\text{span}\{f \in L^2(X) : f \text{ is an eigenfunction}\}}.$$

Show that if $f \in \mathcal{H}_c$, then the closure of the orbit $\{U_T^n f : n \in \mathbb{N}\}$ in the $L^2(X)$ -norm is compact (for this reason, functions in $\mathcal{H}_c$ are called compact functions or almost-periodic functions).

Hint: By completeness, it suffices to show that the orbit $\{U_T^n f : n \in \mathbb{N}\}$ is totally-bounded: for any $\varepsilon > 0$, there exists a finite collection of functions in $g_1, \dots, g_m \in L^2(X)$, such that

$$\min_{1 \leq i \leq m} \|U_T^n f - g_i\|_{L^2(X)} < \varepsilon.$$

Let $f \in \mathcal{H}_c$ and we want to show that the orbit closure $\overline{\{U_T^n f : n \in \mathbb{N}\}}$ is compact in the topology induced by the L^2 -norm. Following the hint, it suffices to show that the set $\{U_T^n f : n \in \mathbb{N}\}$ is totally bounded.

First of all, assume that f is a finite linear combination of eigenfunctions:

$$f = \sum_{i=1}^k c_i h_i,$$

where $c_i \in \mathbb{R}$ and h_i are eigenfunctions with $U_T h_i = \lambda_i h_i$. We deduce that

$$U_T^n f = \sum_{i=1}^k c_i \lambda_i^n h_i$$

for all $n \in \mathbb{N}$. We set $C = |c_1| \|h_1\|_{L^2(X)} + \dots + |c_k| \|h_k\|_{L^2(X)}$.

Let $\varepsilon > 0$. We consider an ε/C -dense set of points $\mathcal{A}_\varepsilon$ on the unit circle $\mathbb{S}^1$, that is a set such that for any point $z \in \mathbb{T}$, there exists a point on $\mathcal{A}_\varepsilon$, whose distance from z (in the complex plane) is at most ε/C (to construct such a set, take the L -th roots of unity for some $L \gg C\varepsilon^{-1}$).

We also consider the set $\mathcal{A}_\varepsilon^k$ and observe that for any point $z \in \mathbb{S}_1^k$, there exists a point $u \in \mathcal{A}_\varepsilon$, such that $\|z - u\| \leq \varepsilon/C$, where we define the norm on $\mathbb{C}^k$ by

$$\|(z_1, \dots, z_k) - (u_1, \dots, u_k)\| = \max\{|z_1 - u_1|, \dots, |z_k - u_k|\}.$$

For every point $u = (u_1, \dots, u_k)$ in $\mathcal{A}_\varepsilon^k$, we consider the functions

$$g_u = \sum_{i=1}^k c_i u_i h_i.$$

We will prove that the balls with radius ε centered on these functions cover the orbit of f . Indeed, let $n \in \mathbb{N}$. There exists $u \in \mathcal{A}_\varepsilon^k$ depending on n , such that

$$\max\{|\lambda_1^n - u_1|, \dots, |\lambda_k^n - u_k|\} < \frac{\varepsilon}{C}.$$

Using the triangle inequality, we have

$$\|U_T^n f - g_u\|_{L^2(X)} \leq \sum_{i=1}^k \|c_i \lambda_i^n h_i - c_i u_i h_i\|_{L^2(X)} = \sum_{i=1}^k |c_i \lambda_i^n - c_i u_i| \|g_i\|_{L^2(X)} \leq \frac{\varepsilon}{C} \sum_{i=1}^K |c_i| \|g_i\|_{L^2(X)} = \varepsilon.$$

Now assume that we have an arbitrary function in $\mathcal{H}_c$. We deduce that there exists a function f_ε that is a finite linear combination of eigenfunctions, such that $\|f - f_\varepsilon\|_{L^2(X)} < \varepsilon/2$. By the previous argument, there exist functions $g_1, \dots, g_m \in L^2(\mu)$, such that

$$\min_{1 \leq i \leq m} \|U_T^n f_\varepsilon - g_i\| < \varepsilon/2.$$

Using the triangle inequality, we deduce that

$$\min_{1 \leq i \leq m} \|U_T^n f - g_i\|_{L^2(X)} \leq \|U_T^n f - U_T^n f_\varepsilon\|_{L^2(X)} + \min_{1 \leq i \leq m} \|U_T^n f_\varepsilon - g_i\|_{L^2(X)} < \varepsilon.$$

The conclusion follows.